

# MATHEMATICAL RESULTS FOR SOME $\alpha$ MODELS OF TURBULENCE WITH CRITICAL AND SUBCRITICAL REGULARIZATIONS

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**ABSTRACT.** In this paper, we establish the existence of a unique “regular” weak solution to turbulent flows governed by a general family of  $\alpha$  models with critical regularizations. In particular this family contains the simplified Bardina model and the modified Leray- $\alpha$  model. When the regularizations are subcritical, we prove the existence of weak solutions and we establish an upper bound on the Hausdorff dimension of the time singular set of those weak solutions. The result is an interpolation between the bound proved by Scheffer for the Navier-Stokes equations and the regularity result in the critical case.

## 1. INTRODUCTION

Let  $\mathbb{T}_3$  be the three dimensional torus  $\mathbb{T}_3 = (\mathbb{R}^3 / \mathcal{T}_3)$  where  $\mathcal{T}_3 = 2\pi\mathbb{Z}^3 / L$ ,  $L > 0$ ,  $0 \leq \theta_1, \theta_2 \leq 1$ , and  $T \in (0, \infty)$ . Our goal is to prove, for a given  $\mathbf{f} : (0, T) \times \mathbb{T}_3 \rightarrow \mathbb{R}^3$ , the existence of  $(\mathbf{v}, p) : (0, T) \times \mathbb{T}_3 \rightarrow \mathbb{R}^3 \times \mathbb{R}$  which solves in a certain sense the following problem  $\mathcal{NS}(\alpha)$

- (1.1)  $\operatorname{div} \mathbf{v} = 0,$
- (1.2)  $\mathbf{v}_{,t} + \operatorname{div}(\tilde{\mathbf{v}} \otimes \bar{\mathbf{v}}) - \nu \Delta \mathbf{v} = -\nabla p + \mathbf{f},$
- (1.3)  $\alpha^{2\theta} (-\Delta)^{\theta_1} \tilde{\mathbf{v}} + \tilde{\mathbf{v}} = \mathbf{v}, \quad \operatorname{div} \tilde{\mathbf{v}} = 0,$
- (1.4)  $\alpha^{2\theta} (-\Delta)^{\theta_2} \bar{\mathbf{v}} + \bar{\mathbf{v}} = \mathbf{v}, \quad \operatorname{div} \bar{\mathbf{v}} = 0.$

considered in  $(0, T) \times \mathbb{T}_3$  and completed by appropriate boundary and initial conditions. Here,  $\mathbf{v}$  is the fluid velocity field,  $p$  is the pressure,  $\mathbf{f}$  is the external body forces,  $\nu$  stands for the viscosity.

The nonlocal operator  $(-\Delta)^{\theta_i}$ ,  $i = 1, 2$  is defined through the Fourier transform

$$(1.5) \quad \widehat{(-\Delta)^{\theta_i} \mathbf{v}}(\mathbf{k}) = |\mathbf{k}|^{2\theta_i} \widehat{\mathbf{v}}(\mathbf{k}).$$

Fractionnal order Laplace operator has been used in another  $\alpha$  models of turbulence in [18, 3, 10]. Existence and uniqueness of solutions of other modifications of the Navier-Stokes equations have been studied by Ladyzhenskaya [12] Lions [16], Málek et al. [17].

Our task is to find the critical relation between the regularizations  $\theta_1$  and  $\theta_2$  (see Theorem 3.1) needed to establish global in time existence of a unique weak solution to eqs. (1.1)–(1.4) and fulfilling the requirements:

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$(\mathbf{v}, p)$  are spatially periodic with period  $L$ ,

$$(1.6) \quad \int_{\mathbb{T}_3} \mathbf{v}(t, \mathbf{x}) d\mathbf{x} = 0 \quad \text{and} \quad \int_{\mathbb{T}_3} p(t, \mathbf{x}) d\mathbf{x} = 0 \quad \text{for } t \in [0, T],$$

and

$$(1.7) \quad \mathbf{v}(0, x) = \mathbf{v}_0(x) \quad \text{in } \mathbb{T}_3.$$

Concerning the regularized velocities  $\tilde{\mathbf{v}}$ ,  $\bar{\mathbf{v}}$  we deduce from (1.3) and (1.4) that they verify the same boundary conditions as  $\mathbf{v}$ :

$$(1.8) \quad \tilde{\mathbf{v}}(t, \mathbf{x} + L\mathbf{e}_j) = \tilde{\mathbf{v}}(t, \mathbf{x}) \quad \text{and} \quad \int_{\mathbb{T}_3} \tilde{\mathbf{v}}(t, \mathbf{x}) d\mathbf{x} = 0 \quad \text{on } (0, T) \times \mathbb{T}_3.$$

$$(1.9) \quad \bar{\mathbf{v}}(t, \mathbf{x} + L\mathbf{e}_j) = \bar{\mathbf{v}}(t, \mathbf{x}) \quad \text{and} \quad \int_{\mathbb{T}_3} \bar{\mathbf{v}}(t, \mathbf{x}) d\mathbf{x} = 0 \quad \text{on } (0, T) \times \mathbb{T}_3,$$

We note that the  $\alpha$  family considered here is a particular case of the general study in [10] where the results do not recover the critical case  $2\theta_1 + \theta_2 = \frac{1}{2}$ . The Leray- $\alpha$  model with critical regularization is studied in [3]. We know, thanks to the works [11, 7] that for  $\theta_1 = 0$ ,  $\theta_2 = 1$  or  $\theta_1 = 1$ ,  $\theta_2 = 0$ , that there exist a unique weak solution to the model (1.1)–(1.4).

When  $\theta_1 = 1$ ,  $\theta_2 = 1$ , we get the simplified Bardina model [7]. The simplified Bardina model first arose in the context of turbulence models for the Navier-Stokes equations in [13]. Based on this work, we will study in a forthcoming paper the model studied in [13, 14] and other related model [8, 4] in the special case where the filtering is given by  $\alpha^{2\theta}(-\Delta)^\theta \bar{\phi} + \bar{\phi} = \phi$ .

When the relation between the regularizations  $\theta_1$  and  $\theta_2$  is subcritical we will prove that  $\frac{1-2\theta_2-4\theta_1}{2}$ -dimensional Hausdorff measure of the time singular set  $\mathcal{S}_{\theta_1, \theta_2}(\bar{\mathbf{v}})$  of any weak solution  $\bar{\mathbf{v}}$  of (1.1)–(1.4) is zero (see Theorem 4.3).

The Hausdorff dimension of the time singular set to weak solutions of another modification of the Navier-Stokes equations was studied in [6, 1].

As a conclusion our study gives the critical regularizations to various  $\alpha$  models, namely the modified Leray- $\alpha$  [11] and the simplified Bardina model [7]. These critical regularisations and the Hausdorff measure of the time singular set in the subcritical case are listed in table 1.

	simplified Bardina	Leray- $\alpha$	modified Leray- $\alpha$
$\theta_1$	$\frac{1}{6}$	$\frac{1}{4}$	0
$\theta_2$	$\frac{1}{6}$	0	$\frac{1}{2}$
$\mathcal{H}(\mathcal{S})$	$\frac{1-6\theta_1}{2}$	$\frac{1-4\theta_1}{2}$	$\frac{1-2\theta_2}{2}$

TABLE 1. Comparison of various critical regularizations and Hausdorff measure for the simplified Bardina, Leray- $\alpha$  and modified Leray- $\alpha$

Observe that the results reported here are also valid in the whole space  $\mathbb{R}^3$  by employing the relevant analogue tools for treating the Navier-Stokes in the whole space [5, 2].

This paper is organized as follows. Section 2 consists of notation and conventions used throughout. In section 3 we prove the global existence and uniqueness of the solution to the model (1.1)–(1.4) with critical regularization. Section 4 treats the question of the subcritical regularizations where we give an upper bound on the Hausdorff dimension of the time singular set of weak solutions to the model (1.1)–(1.4). The result is an interpolation between the bound proved by Scheffer for the Navier-Stokes equations and the regularity result in the critical case.

## 2. NOTATIONS

Before formulating the main results of this paper, we fix notation of function spaces that we shall employ. We denote by  $L^p(\mathbb{T}_3)$  and  $W^{r,p}(\mathbb{T}_3)$ ,  $r \geq -1$ ,  $1 \leq p \leq \infty$ , the usual Lebesgue and Sobolev spaces over  $\mathbb{T}_3$ , and the Bochner spaces  $C(0, T; X)$ ,  $L^p(0, T; X)$  are defined in the standard way.

The Sobolev spaces  $\mathbf{H}^s = H^s(\mathbb{T}_3)^3$ , of mean-free functions are classically characterized in terms of the Fourier series

$$\mathbf{H}^s = \left\{ \mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{T}_3} \mathbf{c}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, (\mathbf{c}_{\mathbf{k}})^* = \mathbf{c}_{-\mathbf{k}}, \mathbf{c}_0 = 0, \|\mathbf{v}\|_{s,2}^2 = \sum_{\mathbf{k} \in \mathbb{T}_3} |\mathbf{k}|^{2s} |\mathbf{c}_{\mathbf{k}}|^2 < \infty \right\},$$

where  $(\mathbf{c}_{\mathbf{k}}^N)^*$  denote the complex conjugate  $\mathbf{c}_{\mathbf{k}}^N$ . In addition we introduce

$$\begin{aligned} \mathbf{H}_{\text{div}}^s &= \{\mathbf{v} \in \mathbf{H}^s; \text{div } \mathbf{v} = 0 \text{ in } \mathbb{T}_3\}, \\ \mathbf{H}^{-s} &= (\mathbf{H}^s)', \quad \mathbf{L}^2 = \mathbf{H}^0, \quad \mathbf{L}_{\text{div}}^2 = \mathbf{H}_{\text{div}}^0. \end{aligned}$$

Let us mention that by using Poincaré inequality we have

$$(2.1) \quad \|\mathbf{v}\|_{s,2} \approx \|\tilde{\mathbf{v}}\|_{s+2\theta_1,2} \approx \|\bar{\mathbf{v}}\|_{s+2\theta_2,2}.$$

Throughout we will use  $C$  to denote an arbitrary constant which may change line to line.

## 3. EXISTENCE AND UNIQUENESS IN THE CRITICAL CASE: $2\theta_1 + \theta_2 = \frac{1}{2}$

The aim in this section is to find the critical relation between  $\theta_1$  and  $\theta_2$  that ensures the existence and the uniqueness of the weak solution to the model (1.1)–(1.4).

**Theorem 3.1.** *Assume that  $2\theta_1 + \theta_2 = \frac{1}{2}$ ,  $0 \leq \theta_1 < \frac{1}{4}$  and  $0 < \theta_2 \leq \frac{1}{2}$ . Let  $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1})$  be a divergence free function and  $\mathbf{v}_0 \in L_{\text{div}}^2$ . Then there exist  $(\mathbf{v}, p)$  a unique “regular” weak solution to (1.1)–(1.4) such that*

$$(3.1) \quad \mathbf{v} \in C(0, T; \mathbf{L}_{\text{div}}^2) \cap L^2(0, T; \mathbf{H}_{\text{div}}^1),$$

$$(3.2) \quad \mathbf{v}_{,t} \in L^2(0, T; \mathbf{H}^{-1}),$$

$$(3.3) \quad p \in L^2(0, T; L^2(\mathbb{T}_3)).$$

fulfill

$$(3.4) \quad \begin{aligned} &\int_0^T \langle \mathbf{v}_{,t}, \mathbf{w} \rangle - (\tilde{\mathbf{v}} \otimes \bar{\mathbf{v}}, \nabla \mathbf{w}) + \nu(\nabla \mathbf{v}, \nabla \mathbf{w}) dt - (p, \text{div } \mathbf{w}) \\ &= \int_0^T \langle \mathbf{f}, \mathbf{w} \rangle dt \quad \text{for all } \mathbf{w} \in L^2(0, T; \mathbf{H}^1), \end{aligned}$$

Moreover,

$$(3.5) \quad \mathbf{v}(0) = \mathbf{v}_0.$$

**Remark 3.1.** We use the name “regular” for the weak solution since the weak solution is unique and the velocity part of the solution  $\mathbf{v}$  is a possible test function in the weak formulation (3.4), that in particular implies that  $\mathbf{v} \in C(0, T; \mathbf{L}^2_{\text{div}})$ .

**Remark 3.2.** Once existence and uniqueness in the large of a weak solution to the model (1.1)–(1.4) with critical regularization is known. Further theoretical properties of the model can then be developed. These are currently under study by the author and will be presented in a subsequent report.

**Remark 3.3.** In a further paper, the author will prove that the solution  $(\bar{\mathbf{v}}, p)$  of the model (1.1)–(1.4) converges in some sense to a solution of the Navier-Stokes equations when  $\alpha$  goes to zero.

### Proof of Theorem 3.1

The proof of Theorem 3.1 follows the classical scheme. We start by constructing approximated solutions  $(\mathbf{v}^N, p^N)$  via Galerkin method. Then we seek for a priori estimates that are uniform with respect to  $N$ . Next, we pass to the limit in the equations after having used compactness properties. Finally we show that the solution we constructed is unique thanks to Gronwall’s lemma. We also note that in our argument we keep the pressure in the weak formulation of the problem and we do not simply neglect it by projecting the equations over divergence-free vector fields.

**Step 1**(Galerkin approximation). Consider the sequence  $\{e^{i\mathbf{k}\cdot\mathbf{x}}\}_{|\mathbf{k}|=1}^\infty$  consisting of  $L^2$ -orthonormal and  $W^{1,2}$ -orthogonal eigenvectors of the following problem:

$$(3.6) \quad -\Delta e^{i\mathbf{k}\cdot\mathbf{x}} = |\mathbf{k}|^2 e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ in } \mathbb{T}_3, \quad \text{for all } \mathbf{k} \in \mathcal{T}_3 \setminus \{0\}.$$

We note that this sequence forms a hilbertian basis of  $\mathbf{L}^2$ .

We set

$$(3.7) \quad \mathbf{v}^N(t, \mathbf{x}) = \sum_{|\mathbf{k}|=1}^N \mathbf{c}_\mathbf{k}^N(t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

such that  $\mathbf{k} \cdot \mathbf{c}_\mathbf{k}^N = 0$  for all  $\mathbf{k} \in \mathcal{T}_3 \setminus \{0\}$  and  $(\mathbf{c}_\mathbf{k}^N)^* = \mathbf{c}_{-\mathbf{k}}^N$ . Thus due of (1.3) and (1.4) we have

$$(3.8) \quad \tilde{\mathbf{v}}^N(t, \mathbf{x}) = \sum_{|\mathbf{k}|=1}^N \tilde{\mathbf{c}}_\mathbf{k}^N(t) e^{i\mathbf{k}\cdot\mathbf{x}} \text{ and } \bar{\mathbf{v}}^N(t, \mathbf{x}) = \sum_{|\mathbf{k}|=1}^N \bar{\mathbf{c}}_\mathbf{k}^N(t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

where

$$(3.9) \quad \tilde{\mathbf{c}}_\mathbf{k}^N = \frac{\mathbf{c}_\mathbf{k}^N}{1 + \alpha^{2\theta_1} |\mathbf{k}|^{2\theta_1}} \text{ and } \bar{\mathbf{c}}_\mathbf{k}^N = \frac{\mathbf{c}_\mathbf{k}^N}{1 + \alpha^{2\theta_2} |\mathbf{k}|^{2\theta_2}},$$

for all  $\mathbf{k} \in \mathcal{T}_3 \setminus \{0\}$ .

We look for  $(\mathbf{v}^N(t, \mathbf{x}), p^N(t, \mathbf{x}))$  that are determined through the system of equations

$$(3.10) \quad (\mathbf{v}_{,t}^N, e^{i\mathbf{k}\cdot\mathbf{x}}) - (\tilde{\mathbf{v}}^N \otimes \bar{\mathbf{v}}^N, \nabla e^{i\mathbf{k}\cdot\mathbf{x}}) + \nu(\nabla \mathbf{v}^N, \nabla e^{i\mathbf{k}\cdot\mathbf{x}}) - (p^N, \operatorname{div} e^{i\mathbf{k}\cdot\mathbf{x}}) = \langle \mathbf{f}, e^{i\mathbf{k}\cdot\mathbf{x}} \rangle, \quad |\mathbf{k}| = 1, 2, \dots, N,$$

and

$$(3.11) \quad p^N = - \sum_{i,j} \partial_i \partial_j \Delta^{-1} (\Pi^N(\tilde{v}_i^N \bar{v}_j^N)) = - \sum_{i,j} R_{ij} (\Pi^N(\tilde{v}_i^N \bar{v}_j^N)).$$

Where the projector  $\Pi^N$  assign to any Fourier series  $\sum_{\mathbf{k} \in \mathcal{T}_3 \setminus \{0\}} \mathbf{g}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$  its N-dimensional part, i.e.  $\sum_{\mathbf{k} \in \mathcal{T}_3 \setminus \{0\}, |\mathbf{k}| \leq N} \mathbf{g}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$ , and  $R_{ij}$  is the Riez operator defined through the Fourier transform by

$$(3.12) \quad \widehat{R_{ij}(u)} = \frac{k_i k_j}{|\mathbf{k}|^2} \widehat{u(\mathbf{k})}, \quad \text{for all } \mathbf{k} \in \mathcal{T}_3 \setminus \{0\}.$$

Moreover we require that  $\mathbf{v}^N$  satisfies the following initial condition

$$(3.13) \quad \mathbf{v}^N(0, \cdot) = \mathbf{v}_0^N = \sum_{|\mathbf{k}|=1}^N \mathbf{c}_0^N e^{i\mathbf{k}\cdot\mathbf{x}},$$

and

$$(3.14) \quad \mathbf{v}_0^N \rightarrow \mathbf{v}_0 \quad \text{strongly in } L^2(0, T; \mathbf{L}^2) \quad \text{when } N \rightarrow \infty.$$

Where the initial condition  $\bar{\mathbf{v}}_0^N$  is deduced from  $\mathbf{v}_0^N$  through the relation (1.4).

The classical Caratheodory theory [24] then implies the short-time existence of solutions to (3.10)-(3.11). Next we derive estimate on  $\mathbf{c}^N$  that is uniform w.r.t.  $N$ . These estimates then imply that the solution of (3.10)-(3.11) constructed on a short time interval  $[0, T^N[$  exists for all  $t \in [0, T]$ .

**Step 2** (Uniform estimates 1) Multiplying the  $|\mathbf{k}|$ th equation in (3.10) with  $\bar{\mathbf{c}}_{\mathbf{k}}^N(t)$ , summing over  $|\mathbf{k}| = 1, 2, \dots, N$ , integrating over time from 0 to  $t$  and using the following identities

$$(3.15) \quad (\mathbf{v}_{,t}^N, \bar{\mathbf{v}}^N) = (\bar{\mathbf{v}}_{,t}^N + \alpha^{2\theta} (-\Delta)^{\theta} \bar{\mathbf{v}}_{,t}^N, \bar{\mathbf{v}}^N) = \frac{1}{2} \frac{d}{dt} \|\bar{\mathbf{v}}^N\|_2^2 + \frac{1}{2} \frac{d}{dt} \|\bar{\mathbf{v}}^N\|_{\theta,2}^2,$$

$$(3.16) \quad (-\Delta \mathbf{v}^N, \bar{\mathbf{v}}^N) = (-\Delta \bar{\mathbf{v}}^N - \alpha^{2\theta} \Delta (-\Delta)^{\theta} \bar{\mathbf{v}}^N, \bar{\mathbf{v}}^N) = \|\bar{\mathbf{v}}^N\|_{1,2}^2 + \|\bar{\mathbf{v}}^N\|_{1+\theta,2}^2,$$

and

$$(3.17) \quad (\tilde{\mathbf{v}}^N \otimes \bar{\mathbf{v}}^N, \nabla \bar{\mathbf{v}}^N) = \left( \tilde{\mathbf{v}}^N, \nabla \frac{|\bar{\mathbf{v}}^N|^2}{2} \right) = - \left( \operatorname{div} \tilde{\mathbf{v}}^N, \frac{|\bar{\mathbf{v}}^N|^2}{2} \right) = 0$$

leads to the a priori estimates

$$(3.18) \quad \begin{aligned} & \frac{1}{2} (\|\bar{\mathbf{v}}^N\|_2^2 + \|\bar{\mathbf{v}}^N\|_{\theta,2}^2) + \nu \int_0^t (\|\bar{\mathbf{v}}^N\|_{1,2}^2 + \|\bar{\mathbf{v}}^N\|_{1+\theta,2}^2) ds \\ &= \int_0^t \langle \mathbf{f}, \bar{\mathbf{v}}^N \rangle ds + \frac{1}{2} (\|\bar{\mathbf{v}}_0\|_2^2 + \|\bar{\mathbf{v}}_0\|_{\theta,2}^2). \end{aligned}$$

Using the duality norm comined with Young inequality we conclude from eqs. (3.18) that

$$(3.19) \quad \sup_{t \in [0, T^N]} \|\bar{\mathbf{v}}^N\|_2^2 + \sup_{t \in [0, T^N]} \|\bar{\mathbf{v}}^N\|_{\theta_2, 2}^2 + \nu \int_0^t (\|\bar{\mathbf{v}}^N\|_{1,2}^2 + \|\bar{\mathbf{v}}^N\|_{1+\theta_2, 2}^2) ds \leq C$$

that immediately implies that the existence time is independent of  $N$  and it is possible to take  $T = T^N$ .

We deduce from 3.19 that

$$(3.20) \quad \bar{\mathbf{v}}^N \in L^\infty(0, T; \mathbf{H}_{\text{div}}^{\theta_2}) \cap L^2(0, T; \mathbf{H}^{1+\theta_2}(\mathbb{T}_3)^3),$$

thus from the relation (1.4) combined with the Poincaré inequality we conclude that

$$(3.21) \quad \mathbf{v}^N \in L^\infty(0, T; \mathbf{H}_{\text{div}}^{-\theta_2}) \cap L^2(0, T; \mathbf{H}^{1-\theta_2}).$$

From (1.3) it follows that

$$(3.22) \quad \tilde{\mathbf{v}}^N \in L^\infty(0, T; \mathbf{H}_{\text{div}}^{2\theta_1-\theta_2}) \cap L^2(0, T; \mathbf{H}^{1+2\theta_1-\theta_2}).$$

**Step 3** (Uniform estimates 2) Let us come back to the relation (3.10), multiplying the  $|\mathbf{k}|$ th equation in (3.10) with  $\mathbf{c}_k^N(t)$ , summing over  $|\mathbf{k}| = 1, 2, \dots, N$ , we conclude that

$$(3.23) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{v}^N\|_2^2 + \nu \|\mathbf{v}^N\|_{1,2}^2 \leq \left| \left( \tilde{\mathbf{v}}^N \otimes \bar{\mathbf{v}}^N, \nabla \mathbf{v}^N \right) \right| + \langle \mathbf{f}, \mathbf{v}^N \rangle := I_1 + I_2.$$

For  $I_1$  we have for  $\frac{1}{2} - 2\theta_1 + \theta_2 \leq 2\theta_2$  i.e.  $2\theta_1 + \theta_2 \geq \frac{1}{2}$  that

$$(3.24) \quad \begin{aligned} I_1 &\leq \|\tilde{\mathbf{v}}^N \otimes \bar{\mathbf{v}}^N\|_2 \|\nabla \mathbf{v}^N\|_2 \\ &\leq \frac{C}{\nu} \|\tilde{\mathbf{v}}^N \otimes \bar{\mathbf{v}}^N\|_2^2 + \frac{\nu}{4} \|\nabla \mathbf{v}^N\|_2^2 \\ &\leq \frac{C}{\nu} \|\tilde{\mathbf{v}}^N\|_{1+2\theta_1-\theta_2, 2}^2 \|\bar{\mathbf{v}}^N\|_{\frac{1}{2}-2\theta_1+\theta_2, 2}^2 + \frac{\nu}{4} \|\nabla \mathbf{v}^N\|_2^2 \\ &\leq \frac{C}{\nu} \|\tilde{\mathbf{v}}^N\|_{1+2\theta_1-\theta_2, 2}^2 \|\bar{\mathbf{v}}^N\|_{2\theta_2, 2}^2 + \frac{\nu}{4} \|\nabla \mathbf{v}^N\|_2^2. \end{aligned}$$

Now we use the following inequality (see in [3].)

$$(3.25) \quad \|\bar{\mathbf{v}}^N\|_{2\theta_2, 2}^2 \leq \frac{1}{\alpha^{2\theta_2}} \|\mathbf{v}^N\|_2^2.$$

We conclude that

$$(3.26) \quad I_1 \leq \frac{C}{\nu} \frac{1}{\alpha^{2\theta}} \|\mathbf{v}^N\|_2^2 \|\tilde{\mathbf{v}}^N\|_{1+2\theta_1-\theta_2, 2}^2 + \frac{\nu}{4} \|\nabla \mathbf{v}^N\|_2^2.$$

To estimate  $I_2$  we use the duality norm and Young inequality in order to obtain

$$(3.27) \quad I_2 \leq \|\mathbf{f}\|_{-1,2} \|\mathbf{v}^N\|_{1,2} \leq \frac{C}{\nu} \|\mathbf{f}\|_{-1,2}^2 + \frac{\nu}{4} \|\mathbf{v}^N\|_{1,2}^2.$$

Thus (3.26) and (3.27) lead to the conclusion that

$$(3.28) \quad \frac{1}{2} \frac{d}{dt} \|\mathbf{v}^N\|_2^2 + \nu \|\mathbf{v}^N\|_{1,2}^2 \leq \frac{C}{\nu} \frac{1}{\alpha^{2\theta}} \|\mathbf{v}^N\|_2^2 \|\tilde{\mathbf{v}}^N\|_{1+2\theta_1-\theta_2, 2}^2 + \frac{C}{\nu} \|\mathbf{f}\|_{-1,2}^2.$$

Integrating (3.28) over time from 0 to  $T$  and using Gronwall's Lemma and (3.22) lead to the following estimate

$$(3.29) \quad \sup_{t \in [0, T]} \|\mathbf{v}^N\|_2^2 + \nu \int_0^T \|\mathbf{v}^N\|_{1,2}^2 dt \leq C.$$

We deduce from (3.29) that

$$(3.30) \quad \mathbf{v}^N \in L^\infty(0, T; \mathbf{L}_{\text{div}}^2) \cap L^2(0, T; \mathbf{H}_{\text{div}}^1),$$

thus from the relation (1.4) we conclude that

$$(3.31) \quad \tilde{\mathbf{v}}^N \in L^\infty(0, T; \mathbf{H}_{\text{div}}^{2\theta_2}) \cap L^2(0, T; \mathbf{H}_{\text{div}}^{1+2\theta_2}),$$

and from (1.3) we obtain

$$(3.32) \quad \bar{\mathbf{v}}^N \in L^\infty(0, T; \mathbf{H}_{\text{div}}^{2\theta_1}) \cap L^2(0, T; \mathbf{H}_{\text{div}}^{1+2\theta_1}).$$

We observe from (3.31) and (3.32) that for all  $2\theta_1 + 2\theta_2 \geq \frac{1}{2}$ , in particular for  $2\theta_1 + \theta_2 \geq \frac{1}{2}$ , we have

$$(3.33) \quad \tilde{\mathbf{v}}^N \otimes \bar{\mathbf{v}}^N \in L^2(0, T; L^2(\mathbb{T}_3)^{3 \times 3}).$$

Consequently from the Calderon-Zygmund theory eqs (3.11) implies that

$$(3.34) \quad \int_0^T \|p^N\|_2^2 dt < K.$$

From eqs. (3.10), (3.31) and (3.32) we also obtain that

$$(3.35) \quad \int_0^T \|\mathbf{v}_{,t}^N\|_{-1,2}^2 dt < K.$$

and thus from the relations (1.3) and (1.4) we deduce

$$(3.36) \quad \int_0^T \|\tilde{\mathbf{v}}_{,t}^N\|_{-1,2}^2 dt < K, \quad \text{and} \quad \int_0^T \|\bar{\mathbf{v}}_{,t}^N\|_{-1,2}^2 dt \leq K.$$

**Step 4** (Limit  $N \rightarrow \infty$ ) It follows from the estimates (3.30)-(3.36) and the Aubin-Lions compactness lemma (see [21] for example) that there are a not relabeled subsequence of  $(\mathbf{v}^N, \tilde{\mathbf{v}}^N, \bar{\mathbf{v}}^N, p^N)$  and a quadruplet  $(\mathbf{v}, \tilde{\mathbf{v}}, \bar{\mathbf{v}}, p)$  such that

- |        |   |   |
|--------|---|---|
| (3.37) | $\mathbf{v}^N \rightharpoonup^* \mathbf{v}$                         | weakly* in $L^\infty(0, T; \mathbf{L}^2)$ ,           |
| (3.38) | $\tilde{\mathbf{v}}^N \rightharpoonup^* \tilde{\mathbf{v}}$         | weakly* in $L^\infty(0, T; \mathbf{H}^{2\theta_1})$ , |
| (3.39) | $\bar{\mathbf{v}}^N \rightharpoonup^* \bar{\mathbf{v}}$             | weakly* in $L^\infty(0, T; \mathbf{H}^{2\theta_2})$ , |
| (3.40) | $\mathbf{v}^N \rightharpoonup \mathbf{v}$                           | weakly in $L^2(0, T; \mathbf{H}^1)$ ,                 |
| (3.41) | $\tilde{\mathbf{v}}^N \rightharpoonup \tilde{\mathbf{v}}$           | weakly in $L^2(0, T; \mathbf{H}^{1+2\theta_1})$ ,     |
| (3.42) | $\bar{\mathbf{v}}^N \rightharpoonup \bar{\mathbf{v}}$               | weakly in $L^2(0, T; \mathbf{H}^{1+2\theta_2})$ ,     |
| (3.43) | $\mathbf{v}_{,t}^N \rightharpoonup \mathbf{v}_{,t}$                 | weakly in $L^2(0, T; \mathbf{H}^{-1})$ ,              |
| (3.44) | $\tilde{\mathbf{v}}_{,t}^N \rightharpoonup \tilde{\mathbf{v}}_{,t}$ | weakly in $L^2(0, T; \mathbf{H}^{-1})$ ,              |
| (3.45) | $\bar{\mathbf{v}}_{,t}^N \rightharpoonup \bar{\mathbf{v}}_{,t}$     | weakly in $L^2(0, T; \mathbf{H}^{-1})$ ,              |
| (3.46) | $p^N \rightharpoonup p$   | weakly in $L^2(0, T; L^2(\mathbb{T}_3))$ ,            |
| (3.47) | $\mathbf{v}^N \rightarrow \mathbf{v}$                               | strongly in $L^2(0, T; \mathbf{L}^2)$ ,               |
| (3.48) | $\tilde{\mathbf{v}}^N \rightarrow \tilde{\mathbf{v}}$               | strongly in $L^2(0, T; \mathbf{L}^2)$ ,               |
| (3.49) | $\bar{\mathbf{v}}^N \rightarrow \bar{\mathbf{v}}$                   | strongly in $L^2(0, T; \mathbf{L}^2)$ .               |

By a standard interpolation argument we have

$$(3.50) \quad \mathbf{v}^N \in L^{\frac{10}{3}}(0, T; L^{\frac{10}{3}}(\mathbb{T}_3)^3),$$

$$(3.51) \quad \tilde{\mathbf{v}}^N \in L^{\frac{10}{3-4\theta_1}}(0, T; L^{\frac{10}{3-4\theta_1}}(\mathbb{T}_3)^3),$$

$$(3.52) \quad \bar{\mathbf{v}}^N \in L^{\frac{10}{3-4\theta_2}}(0, T; L^{\frac{10}{3-4\theta_2}}(\mathbb{T}_3)^3).$$

Thus from (3.50)-(3.52) and (3.47)-(3.49) we obtain

$$(3.53) \quad \mathbf{v}^N \rightarrow \mathbf{v} \quad \text{strongly in } L^{q_1}(0, T; L^{q_1}(\mathbb{T}_3)^3) \text{ for all } q_1 < \frac{10}{3},$$

$$(3.54) \quad \tilde{\mathbf{v}}^N \rightarrow \tilde{\mathbf{v}} \quad \text{strongly in } L^{q_2}(0, T; L^{q_2}(\mathbb{T}_3)^3) \text{ for all } q_2 < \frac{10}{3-4\theta_1},$$

$$(3.55) \quad \bar{\mathbf{v}}^N \rightarrow \bar{\mathbf{v}} \quad \text{strongly in } L^{q_3}(0, T; L^{q_3}(\mathbb{T}_3)^3) \text{ for all } q_3 < \frac{10}{3-4\theta_2},$$

Since  $q_2 < \frac{10}{3-4\theta_1}$ ,  $q_3 < \frac{10}{3-4\theta_2}$  and  $2\theta_1 + \theta_2 = \frac{1}{2}$  the application of Hölder's inequality implies that

$$(3.56) \quad \tilde{\mathbf{v}} \otimes \bar{\mathbf{v}} \in L^q(0, T; L^q(\mathbb{T}_3)^{3 \times 3}) \text{ where } q \geq 2,$$

The above established convergences are clearly sufficient for taking the limit in (3.10) and for concluding that  $(\mathbf{v}, p)$  satisfy (3.4). Moreover, from (3.40) and (3.43) one we can deduce by a classical argument of J.L. Lions [15] that

$$(3.57) \quad \mathbf{v} \in \mathcal{C}(0, T; \mathbf{L}^2).$$

Furthermore, from the strong continuity of  $\mathbf{v}$  with respect to the time with value in  $L^2$  we deduce that  $\mathbf{v}(0) = \mathbf{v}_0$ .

Let us mention also that  $\bar{\mathbf{v}}$  is a possible test in the weak formulation (3.4). Thus  $\bar{\mathbf{v}}$  verifies for all  $t \in [0, T]$  the following equality

$$(3.58) \quad \begin{aligned} & (\|\bar{\mathbf{v}}(t)\|_2^2 + \|\bar{\mathbf{v}}(t)\|_{\theta_2, 2}^2) + 2\nu \int_0^t (\|\bar{\mathbf{v}}\|_{1,2}^2 + \|\bar{\mathbf{v}}\|_{1+\theta_2, 2}^2) ds \\ &= 2 \int_0^t \langle \mathbf{f}, \bar{\mathbf{v}} \rangle ds + (\|\bar{\mathbf{v}}_0\|_2^2 + \|\bar{\mathbf{v}}_0\|_{\theta_2, 2}^2). \end{aligned}$$

**Step 5** (Uniqueness) Since the pressure part of the solution is uniquely determined by the velocity part it remains to show the uniqueness to the velocity.

Next, we will show the continuous dependence of the solutions on the initial data and in particular the uniqueness.

Let  $(\mathbf{v}_1, p_1)$  and  $(\mathbf{v}_2, p_2)$  any two solutions of (1.1)-(1.4) on the interval  $[0, T]$ , with initial values  $\mathbf{v}_1(0)$  and  $\mathbf{v}_2(0)$ . Let us denote by  $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$ ,  $\tilde{\mathbf{w}} = \tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2$  and  $\bar{\mathbf{w}} = \bar{\mathbf{v}}_1 - \bar{\mathbf{v}}_2$ . We subtract the equation for  $\mathbf{v}_1$  from the equation for  $\mathbf{v}_2$  and test it with  $\mathbf{w}$ .

In the following we distinguish between two cases.

Case 1:  $2\theta_1 + \theta_2 = \frac{1}{2}$ ,  $0 \leq \theta_1 < \frac{1}{4}$  and  $0 < \theta_2 < \frac{1}{2}$ .

We get using successively Cauchy-Schwarz inequality, Young inequality, embedding theorem and the relations (1.3) and (1.4).

$$\begin{aligned}
 \|w_{,t}\|_2^2 + \nu \|\nabla w\|_2^2 &\leq \frac{4}{\nu} \|\tilde{w}\bar{v}_1\|_2^2 + \frac{4}{\nu} \|\tilde{v}_2\bar{w}\|_2^2 \\
 (3.59) \quad &\leq \frac{4}{\nu} \|\tilde{w}\|_{\frac{1}{2}-\theta_2,2}^2 \|\bar{v}_1\|_{1+\theta_2}^2 + \frac{4}{\nu} \|\bar{w}\|_{\frac{1}{2}-2\theta_1,2}^2 \|\tilde{v}_2\|_{1+2\theta_1}^2 \\
 &\leq \frac{1}{\alpha^{2\theta_1+2\theta_2}} \frac{4}{\nu} \|w\|_2^2 (\|v_1\|_{1,2}^2 + \|v_2\|_{1,2}^2).
 \end{aligned}$$

Case 2:  $\theta_1 = 0$  and  $\theta_2 = \frac{1}{2}$ .  
In this case we have that

$$(3.60) \quad \bar{v}^N \in L^\infty(0, T; \mathbf{H}_{\text{div}}^1) \cap L^2(0, T; \mathbf{H}_{\text{div}}^2),$$

We get using successively Cauchy-Schwarz inequality, Young inequality, embedding theorem and the relation (1.4).

$$\begin{aligned}
 \|w_{,t}\|_2^2 + \nu \|\nabla w\|_2^2 &\leq \frac{4}{\nu} \|w\bar{v}_1\|_2^2 + \frac{4}{\nu} \|v_2\bar{w}\|_2^2 \\
 (3.61) \quad &\leq \frac{4}{\nu} \|w\|_2^2 \|\bar{v}_1\|_{\frac{3}{2}+\epsilon}^2 + \frac{4}{\nu} \|\bar{w}\|_{1,2}^2 \|v_2\|_{\frac{1}{2},2}^2 \\
 &\leq \frac{1}{\alpha^2} \frac{4}{\nu} \|w\|_2^2 (\|v_1\|_{2,2}^2 + \|v_2\|_{1,2}^2).
 \end{aligned}$$

Using Gronwall's inequality we conclude the continuous dependence of the solutions on the initial data in the  $L^\infty([0, T], \mathbf{L}^2)$  norm. In particular, if  $w_0 = 0$  then  $w = 0$  and the solutions are unique for all  $t \in [0, T]$ . Since  $T > 0$  is arbitrary this solution may be uniquely extended for all time.

This finish the proof of Theorem 3.1.

#### 4. HAUSDORFF DIMENSION OF THE TIME SINGULAR SET IN THE SUBCRITICAL CASE: $2\theta_1 + \theta_2 < \frac{1}{2}$

The aim in this section is to establish an upper bound for the Hausdorff dimension of the time singular set  $\mathcal{S}_{\theta_1, \theta_2}$  of the solutions  $\bar{v}$  of (1.1)–(1.4), see Theorem 4.3 below. We know, thanks to Scheffer's work [19, 20], that if  $v$  is a weak Leray solution of the Navier-Stokes equations then the  $\frac{1}{2}$ -dimensional Hausdorff measure of the time singular set of  $v$  is zero. Further, when  $2\theta_1 + \theta_2 = \frac{1}{2}$ , we proved in the above section the existence of a unique regular weak solution to the model (1.1)–(1.4). Therefore, it is interesting to understand how the time singular set  $\mathcal{S}_{\theta_1, \theta_2}(\bar{v})$  may depend on the regularization parameters  $\theta_1$  and  $\theta_2$ .

We divide this section into four subsections. One is devoted to prove the existence of weak solutions. The second one is devoted to prove the existence of a unique strong solution. An additional subsection is devoted to the definitions of the Hausdorff dimension and the singular time set. The final subsection is devoted to the proof of Theorem 4.3 which is the main result of this section.

#### 4.1. Existence of weak solutions.

**Theorem 4.1.** Assume that  $2\theta_1 + \theta_2 < \frac{1}{2}$ . Let  $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1})$  be a divergence free function and  $\bar{\mathbf{v}}_0 \in \mathbf{H}_{\text{div}}^{\theta_2}$ . Then for any  $T > 0$  there exist  $(\bar{\mathbf{v}}, \mathbf{v}, p)$  a weak distributional solution to (1.1)–(1.4) such that

$$(4.1) \quad \begin{aligned} \bar{\mathbf{v}} &\in C_{\text{weak}}([0, T]; \mathbf{H}_{\text{div}}^{\theta_2}) \cap L^2([0, T]; \mathbf{H}_{\text{div}}^{1+\theta_2}), \\ \mathbf{v} &\in C_{\text{weak}}([0, T]; \mathbf{H}_{\text{div}}^{-\theta_2}) \cap L^2([0, T]; \mathbf{H}_{\text{div}}^{1-\theta_2}), \end{aligned}$$

$$(4.2) \quad \begin{aligned} \frac{\partial \bar{\mathbf{v}}}{\partial t} &\in L^{\frac{5}{3-2\theta_1-2\theta_2}}([0, T]; W^{-1+2\theta_2, \frac{5}{3-2\theta_1-2\theta_2}}(\mathbb{T}_3)^3)), \\ \frac{\partial \mathbf{v}}{\partial t} &\in L^{\frac{5}{3-2\theta_1-2\theta_2}}([0, T]; W^{-1, \frac{5}{3-2\theta_1-2\theta_2}}(\mathbb{T}_3)^3)), \end{aligned}$$

$$(4.3) \quad p \in L^{\frac{5}{3-2\theta_1-2\theta_2}}([0, T], L^{\frac{5}{3-2\theta_1-2\theta_2}}(\mathbb{T}_3)),$$

$$(4.4) \quad \begin{aligned} &\int_0^T \langle \frac{\partial \mathbf{v}}{\partial t}, \mathbf{w} \rangle - (\tilde{\mathbf{v}} \otimes \bar{\mathbf{v}}, \nabla \mathbf{w}) + \nu(\nabla \mathbf{v}, \nabla \mathbf{w}) - (p, \text{div } \mathbf{w}) dt \\ &= \int_0^T \langle \mathbf{f}, \mathbf{w} \rangle dt \quad \text{for all } \mathbf{w} \in L^{\frac{5}{2+2\theta_1+2\theta_2}}(0, T; W^{1, \frac{5}{2+2\theta_1+2\theta_2}}(\mathbb{T}_3)^3)), \end{aligned}$$

where the velocity  $\mathbf{v}$  verifies

$$(4.5) \quad \sup_{t \in (0, T)} \|\mathbf{v}(t)\|_{-1,2}^2 + \nu \int_0^T \|\mathbf{v}(t)\|_2^2 dt \leq \|\mathbf{v}_0\|_{-1,2}^2 + \int_0^T \langle \mathbf{f}, \bar{\mathbf{v}} \rangle dt,$$

or equivalently  $\bar{\mathbf{v}}$  verifies

$$(4.6) \quad \begin{aligned} &\sup_{t \in (0, T)} (\|\bar{\mathbf{v}}\|_2^2 + \|\bar{\mathbf{v}}\|_{\theta_2,2}^2) + 2\nu \int_0^T (\|\bar{\mathbf{v}}\|_{1,2}^2 + \|\bar{\mathbf{v}}\|_{1+\theta_2,2}^2) dt \\ &\leq 2 \int_0^T \langle \mathbf{f}, \bar{\mathbf{v}} \rangle dt + (\|\bar{\mathbf{v}}_0\|_2^2 + \|\bar{\mathbf{v}}_0\|_{\theta_2,2}^2), \end{aligned}$$

and the initial data is attained in the following sense

$$(4.7) \quad \begin{aligned} \lim_{t \rightarrow 0+} (\|\mathbf{v}(t) - \mathbf{v}_0\|_{-\theta_2,2}^2) &= 0, \\ \lim_{t \rightarrow 0+} (\|\bar{\mathbf{v}}(t) - \bar{\mathbf{v}}_0\|_{\theta_2,2}^2) &= 0. \end{aligned}$$

**Proof of Theorem 4.1** The proof of Theorem 4.1 follows the lines of the proof of the above Theorem the only difference is that  $\bar{\mathbf{v}}$  is not a good test function in the weak formulation and thus by using the weak convergence we get the inequality (4.5) instead of an equality.

It remains to show weak continuity in (4.1) and (4.7). This is standard for Navier Stokes equation, we refer the reader to [22, Lemma 1.4] and we omit more details.

#### 4.2. Strong solution.

**Theorem 4.2.** Let  $\mathbf{f} \in L^2([0, T], \mathbf{L}_{\text{div}}^2)$  and  $\bar{\mathbf{v}}_0 \in \mathbf{H}_{\text{div}}^{1-\theta_2}$ . Assume that  $0 \leq 2\theta_1 + \theta_2 < \frac{1}{2}$ . Then there exists  $T_* := T_*(\bar{\mathbf{v}}_0)$ , determined by (4.12), and there exists a unique strong solution  $\bar{\mathbf{v}}$  to (1.1)–(1.4) on  $[0, T_*[$  satisfying:

$$\bar{\mathbf{v}} \in C([0, T_*[; \mathbf{H}^{\theta_2}) \cap L^2([0, T_*[; \mathbf{H}_{\text{div}}^{1+\theta_2}),$$

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} \in L^2([0, T_*[; \mathbf{L}^2) \quad \text{and} \quad p \in L^2([0, T_*[, W^{1,2}(\mathbb{T}_3)).$$

**Proof of Theorem 4.2** Taking the  $L^2$ -inner product of (1.2) with  $-\Delta \bar{v}$  and integrating by parts. Using the incompressibility of the velocity field and the duality relation combined with Hölder inequality and Sobolev injection, we obtain

$$(4.8) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{v}\|_{1,2}^2 + \|\bar{v}\|_{1+\theta_2,2}^2) + \nu (\|\bar{v}\|_{2,2}^2 + \|\bar{v}\|_{2+\theta_2,2}^2) \\ & \leq \int_{\mathbb{T}^3} |\tilde{v} \cdot \nabla \bar{v} \Delta \bar{v}| dx + \int_{\mathbb{T}^3} |\mathbf{f} \cdot \Delta \bar{v}| dx \\ & \leq \alpha^{-2\theta_1-2\theta_2} \|\bar{v}\|_{1+\theta_2,2} \|\bar{v}\|_{\frac{3}{2}-2\theta_1,2} \|\bar{v}\|_{2+\theta_2,2} + \|\mathbf{f}\|_2 \|\bar{v}\|_{2,2}. \end{aligned}$$

Interpolating between  $\mathbf{H}^{1+\theta_2}$  and  $\mathbf{H}^{2+\theta_2}$  we get

$$(4.9) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{v}\|_{1,2}^2 + \|\bar{v}\|_{1+\theta_2,2}^2) + \nu (\|\bar{v}\|_{2,2}^2 + \|\bar{v}\|_{2+\theta_2,2}^2) \\ & \leq \alpha^{-2\theta_1-2\theta_2} \|\bar{v}\|_{1+\theta_2,2}^{\frac{3}{2}+\theta_2+2\theta_1} \|\bar{v}\|_{2+\theta_2,2}^{\frac{3}{2}-\theta_2+2\theta_1} + \|\mathbf{f}\|_2 \|\bar{v}\|_{2,2}. \end{aligned}$$

Using Young inequality we get

$$(4.10) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{v}\|_{1,2}^2 + \|\bar{v}\|_{1+\theta_2,2}^2) + \nu (\|\bar{v}\|_{2,2}^2 + \|\bar{v}\|_{2+\theta_2,2}^2) \\ & \leq \frac{1}{\nu} \|\mathbf{f}\|_2^2 + C(\alpha, \theta_1, \theta_2) \|\nabla \bar{v}\|_{1+\theta_2}^{\frac{2(3+2\theta_2+4\theta_1)}{1+2\theta_2+4\theta_1}}. \end{aligned}$$

We get a differential inequality

$$(4.11) \quad Y' \leq C(\alpha, \theta_1, \theta_2, \nu, f) Y^\gamma,$$

where

$$Y(t) = 1 + \|\bar{v}\|_{1+\theta_2,2}^2 \text{ and } \gamma = \frac{2(3+2\theta_2+4\theta_1)}{1+2\theta_2+4\theta_1}$$

We conclude that

$$Y(t) \leq \frac{Y(0)}{(1 - 2Y(0)^{\gamma-1} C(\alpha, \theta_1, \theta_2, \nu, f) t)^{\frac{1}{\gamma-1}}}$$

as long as  $t < \frac{1}{2Y(0)^{\gamma-1} C(\alpha, \theta_1, \theta_2, \nu, f)}$ , and thus we obtain

$$\sup_{t \in [0, T_*]} \|\bar{v}\|_{1+\theta_2,2}^2 \leq 2(1 + \|\bar{v}_0\|_{1+\theta_2,2}^2)$$

$$(4.12) \quad \text{for } t \leq T_* := \frac{3}{8C(\alpha, \theta_1, \theta_2, \nu, f)} \frac{1}{(1 + \|\bar{v}_0\|_{1+\theta_2,2}^2)^{\gamma-1}}.$$

Integrating (4.10) with respect to time on  $[0, T_*]$  gives the following estimates

$$\int_0^{T_*} \|\bar{v}(t)\|_{2+\theta_2,2}^2 dt \leq M(T_*),$$

where

$$M(T_*) = \frac{1}{\nu} \left( \|\bar{v}_0\|_{1+\theta_2,2}^2 + \frac{2}{\nu} \int_0^{T_*} \|\mathbf{f}\|_2^2 dt + C(\alpha, \theta_1, \theta_2) [2(1 + \|\bar{v}_0\|_{1+\theta_2,2}^2)]^\gamma \right).$$

**4.3. The Hausdorff dimension and singular set.** The basic facts about Hausdorff measure can be found in [9]. The following defintion can be found in [23]

**Definition 4.1.** Let  $X$  be a metric and let  $a > 0$ . The  $a$ -dimensionnal Hausdorff measure of a subset  $Y$  of  $X$  is

$$\mu_a(Y) = \lim_{\epsilon \searrow 0} \mu_{a,\epsilon}(Y) = \sup_{\epsilon > 0} \mu_{a,\epsilon}(Y)$$

where

$$\mu_{a,\epsilon}(Y) = \inf \sum_j (\text{diameter } B_j)^a,$$

the infimum being taken over all the coverings of  $Y$  by balls  $B_j$  such that  $\text{diameter } B_j \leq \epsilon$ .

**Definition 4.2.** Let  $T > 0$ . We denote by the time singular set of  $\bar{\mathbf{v}}(t)$ , weak solution of (1.1)–(1.4) given by Theorem 4.1, the set of  $t \in [0, T]$  on which  $\bar{\mathbf{v}}(t) \notin \mathbf{H}^{1+\theta_2}(\mathbb{T}_3)^3$ .

**4.4. Dimesion of the time singular set.** The main result of the section is the following theorem.

**Theorem 4.3.** Let  $\bar{\mathbf{v}}$  be any weak Leray solution to (1.1)–(1.4) given by Theorem 4.1 ( We suppose also that the externel force  $\mathbf{f} \in L^2([0, T], \mathbf{L}_{\text{div}}^2)$  ). Then for any  $T > 0$  the  $\frac{1-2\theta_2-4\theta_1}{2}$ -dimensional Hausdorff measure of the time singular set of  $\mathbf{v}$  is zero.

### Proof of Theorem 4.3

**Step 1:**( Structure of the time singularity set) We begin by the following Lemma that characterize the structure of the time singularity set of a weak solution to (1.1)–(1.4).

**Lemma 4.1.** We assume that  $\bar{\mathbf{v}}_0 \in \mathbf{H}_{\text{div}}^{1+\theta_2}$ ,  $\mathbf{f} \in L^2([0, T], \mathbf{L}_{\text{div}}^2)$  and  $\bar{\mathbf{v}}$  is any weak solution to (1.1)–(1.4) given by Theorem 4.1. Then there exist an open set  $\mathcal{O}$  of  $(0, T)$  such that:

- (i) For all  $t \in \mathcal{O}$  there exist  $t \in (t_1, t_2) \subseteq (0, T)$  such that  $\mathbf{v} \in C((t_1, t_2), \mathbf{H}^{1+\theta_2})$ .
- (ii) The Lebesgue measure of  $[0, T]/\mathcal{O}$  is zero.

**Proof of Lemma 4.1.** Since  $\bar{\mathbf{v}} \in C_{\text{weak}}([0, T]; \mathbf{H}^{1+\theta_2})$ ,  $\bar{\mathbf{v}}(t)$  is well defined for every  $t$  and we can define

$$\begin{aligned} \Sigma &= \{t \in [0, T], \bar{\mathbf{v}}(t) \in \mathbf{H}^{1+\theta_2}\}, \\ \Sigma^c &= \{t \in [0, T], \bar{\mathbf{v}}(t) \notin \mathbf{H}^{1+\theta_2}\}, \\ \mathcal{O} &= \{t \in (0, T), \exists \epsilon > 0, \bar{\mathbf{v}} \in C((t - \epsilon, t + \epsilon), \mathbf{H}^{1+\theta_2})\}. \end{aligned}$$

It is clear that  $\mathcal{O}$  is open.

Since  $\bar{\mathbf{v}} \in L^2([0, T]; \mathbf{H}^{1+\theta_2})$ ,  $\Sigma^c$  has Lebesgue measure zero. Let us take  $t_0$  such that  $t_0 \in \Sigma$ , and  $t_0 \notin \mathcal{O}$ , then according to Theorem 4.2, their exists  $\epsilon > 0$  such that  $\bar{\mathbf{v}} \in C((t_0, t_0 + \epsilon), \mathbf{H}^{1+\theta_2})$ . So,  $t_0$  is the left end of one of the connected components of  $\mathcal{O}$ . Thus  $\Sigma/\mathcal{O}$  is countable and  $[0, T]/\mathcal{O}$  has Lebesgue measure zero. This finishes the proof of Lemma 4.1.

**Remark 4.1.** We deduce from Theorem 4.2 that, if  $(\alpha_i, \beta_i)$ ,  $i \in I$ , is one of the connected components of  $\mathcal{O}$ , then

$$\lim_{t \rightarrow \beta_i} \|\bar{\mathbf{v}}(t)\|_{1+\theta_2,2} = +\infty.$$

Indeed, otherwise Theorem 4.2 would show that there exist an  $\epsilon > 0$  such that  $\bar{\mathbf{v}} \in C((\beta_i, \beta_i + \epsilon), \mathbf{H}^{1+\theta_2})$  and  $\beta_i$  would not be the end of an connected components of  $\mathcal{O}$ .

**Step 2:**(Main estimate) We have the following Lemma:

**Lemma 4.2.** Under the same notations of Lemma 4.1 Let  $(\alpha_i, \beta_i)$ ,  $i \in I$ , be the connected components of  $\mathcal{O}$ . Then

$$(4.13) \quad \sum_{i \in I} (\beta_i - \alpha_i)^{\frac{1-2\theta_2-4\theta_1}{2}} < \infty$$

**Proof of Lemma 4.2.** Let  $(\alpha_i, \beta_i)$  be one of these connected components and let  $t \in (\alpha_i, \beta_i) \subseteq \mathcal{O}$ . Since  $\bar{\mathbf{v}} \in C_{weak}([0, T]; \mathbf{H}^{\theta_2}) \cap L^2([0, T]; \mathbf{H}^{1+\theta_2})$ ,  $\bar{\mathbf{v}}(t)$  is well defined for every  $t \in (\alpha_i, \beta_i)$  and  $t$  can be chosen such that  $\bar{\mathbf{v}}(t) \in \mathbf{H}^{1+\theta_2}$ . According to Theorem 4.2, inequality (4.12), and since  $\|\bar{\mathbf{v}}(\beta_i)\|_{1+\theta_2} = +\infty$ , for  $t \in (\alpha_i, \beta_i)$  we have,

$$\beta_i - t \geq \frac{1}{C(\alpha, \theta_1, \theta_2, \nu, f)} \frac{1}{(1 + \|\bar{\mathbf{v}}(t)\|_{1+\theta_2,2}^2)^{\gamma-1}},$$

where we have used that  $\gamma = \frac{3+2\theta_2+4\theta_1}{1+2\theta_2+4\theta_1} > 1$ .  
Thus

$$\frac{C(\alpha, \theta_1, \theta_2, \nu, f)}{(\beta_i - t)^{\frac{1}{\gamma-1}}} \leq 1 + \|\bar{\mathbf{v}}(t)\|_{1+\theta_2,2}^2.$$

Then we integrate on  $(\alpha_i, \beta_i)$  to obtain

$$C(\alpha, \theta_1, \theta_2, \nu, f) (\beta_i - \alpha_i)^{\frac{-1}{\gamma-1}+1} \leq (\beta_i - \alpha_i) + \int_{\alpha_i}^{\beta_i} \|\bar{\mathbf{v}}(t)\|_{1+\theta_2,2}^2 dt,$$

Adding all these relations for  $i \in I$  we obtain

$$C(\alpha, \theta_1, \theta_2, \nu, f) \sum_{i \in I} (\beta_i - \alpha_i)^{\frac{-1}{\gamma-1}+1} \leq T + \int_0^T \|\bar{\mathbf{v}}(t)\|_{1+\theta_2,2}^2 dt.$$

This finishes the proof of Lemma 4.2.

**Step 3:**(Recovering argument) We set  $\mathcal{S} = \mathcal{S}_{\theta_1, \theta_2}(\bar{\mathbf{v}}) = [0, T] \setminus \mathcal{O}$ . We have to prove that the  $\frac{1-2\theta_2-4\theta_1}{2}$ -dimensional Hausdorff measure of  $\mathcal{S}$  is zero. Since the Lebesgue measure of  $\mathcal{O}$  is finite ,i.e.

$$(4.14) \quad \sum_{i \in I} (\beta_i - \alpha_i) < \infty,$$

it follows from Lemma 4.2 that for every  $\epsilon > 0$  there exist a finite part  $I_\epsilon \subset I$  such that

$$(4.15) \quad \sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i) \leq \epsilon$$

and

$$(4.16) \quad \sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i)^{\frac{1-2\theta_2-4\theta_1}{2}} \leq \epsilon$$

Note that  $\mathcal{S} \subset [0, T] \setminus \bigcup_{i \in I_\epsilon} (\alpha_i, \beta_i)$  and the set  $[0, T] \setminus \bigcup_{i \in I_\epsilon} (\alpha_i, \beta_i)$  is the union of finite number of mutually disjoint closed intervals, say  $B_j$ , for  $j = 1, \dots, N$ . Our aim now is to show that the diameter  $B_j \leq \epsilon$ . Since the intervals  $(\alpha_i, \beta_i)$  are mutually disjoint, each interval  $(\alpha_i, \beta_i)$ ,  $i \in I \setminus I_\epsilon$ , is included in one, and only one, interval  $B_j$ . We denote by  $I_j$  the set of indice  $i$  such that  $(\alpha_i, \beta_i) \subset B_j$ . It is clear that  $I_\epsilon, I_1, \dots, I_N$  is a partition of  $I$  and we have  $B_j = (\bigcup_{i \in I_j} (\alpha_i, \beta_i)) \cup (B_j \cap \mathcal{S})$  for all  $j = 1, \dots, N$ . It follows from (4.14) that

$$(4.17) \quad \text{diameter } B_j = \sum_{i \in I_j} (\beta_i - \alpha_i) \leq \epsilon.$$

Finally in virtue of the definition 4.1 and estimates (4.17), (4.16) and since  $l^\delta \hookrightarrow l^1$  for all  $0 < \delta < 1$  we have

$$(4.18) \quad \begin{aligned} \mu_{\frac{1-2\theta_2-4\theta_1}{2}, \epsilon}(\mathcal{S}) &\leq \sum_{j=1}^N (\text{diameter } B_j)^{\frac{1-2\theta_2-4\theta_1}{2}} \\ &\leq \sum_{j=1}^N \left( \sum_{i \in I_j} (\beta_i - \alpha_i) \right)^{\frac{1-2\theta_2-4\theta_1}{2}} \\ &\leq \sum_{j=1}^N \sum_{i \in I_j} (\beta_i - \alpha_i)^{\frac{1-2\theta_2-4\theta_1}{2}} \\ &= \sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i)^{\frac{1-2\theta_2-4\theta_1}{2}} \leq \epsilon. \end{aligned}$$

Letting  $\epsilon \rightarrow 0$ , we find  $\mu_{\frac{1-2\theta_2-4\theta_1}{2}}(\mathcal{S}) = 0$  and this completes the proof.

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